Linearizable cellular automata

2007 J. Phys. A: Math. Theor. 40 7159

(http://iopscience.iop.org/1751-8121/40/26/004)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 7159–7174 doi:10.1088/1751-8113/40/26/004

Linearizable cellular automata

Atsushi Nobe[1] and Fumitaka Yura[2]

[1] Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama-cho Toyonaka, Osaka 560-8531, Japan
[2] School of Systems Information Science, Future University-Hakodate, 116-2, Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan

E-mail: nobe@sigmath.es.osaka-u.ac.jp

Received 25 January 2007, in final form 4 May 2007
Published 12 June 2007
Online at stacks.iop.org/JPhysA/40/7159

Abstract
The initial value problem for a class of reversible elementary cellular automata with periodic boundaries is reduced to an initial-boundary value problem for a class of linear systems on a finite commutative ring $\mathbb{Z}_2$. Moreover, a family of such linearizable cellular automata is given.

PACS numbers: 02.30.Ik, 05.45.Yv, 87.17.−d

1. Introduction

A cellular automaton is a discrete dynamical system composed of regular array of cells. Each cell takes a finite number of states which are updated by a local transition function in discrete time steps. The updating rule is quite simple, but cellular automata show complicated behaviours. In the 1980s, Wolfram suggested some connections between cellular automata and differential equations and phenomenologically classified them into four classes [1–3]. Wolfram *et al* investigated a mathematical structure of some kinds of cellular automata such as additive [4] or totalistic [1, 2] ones, which have a simple structure arising from their local transitions. In 1989, Takesue introduced elementary reversible cellular automata (ERCA) and a notion of additive conserved quantities [5]. He obtained many additive conserved quantities of ERCA and pointed out the importance of studying additive conserved quantities from the physical point of view. Moreover, Hattori and Takesue studied the general existence condition of the additive conserved quantities [6]. In 1990, Takahashi and Satsuma proposed a soliton cellular automaton (the box–ball system) as an ultimately discretized soliton system [7]. In 1996, Tokihiro *et al* established a direct connection between the box–ball system and the KdV equation by using a procedure called ultradiscretization [8]. Using the ultradiscretization procedure, many properties of the box–ball systems such as N-soliton solutions, bilinear structure and conserved quantities have been investigated thoroughly [9–14], and the box–ball systems are now considered to be integrable. Then, it is natural for us to put a question: are there any integrable cellular automata other than the box–ball systems? The authors at first tried to find reversible cellular automata, because reversibility is a necessary condition for integrability. In the precedent paper [15], we introduced a graph-theoretical criterion for reversibility of cellular automata and classified all reversible elementary cellular automata (ECA) [1] with periodic boundaries. Some of the reversible ECA are trivial or additive; however, there exist some nonlinear reversible cellular automata whose mathematical structure has not been studied in detail yet.

In this paper, we show that some of the nonlinear reversible ECA can be reduced to some linear systems on a finite commutative ring $\mathbb{Z}_2$ by properly dividing their initial configurations. It should be noted that the box–ball system with periodic boundaries can also be reduced to a linear system on a finite set, which is an ultradiscrete analogue of the Jacobi variety [16, 17]. Then we generalize the linearizable ECA to cellular automata which take arbitrary number of states and whose time evolution depends on arbitrary size of the neighbourhood. A family of linearizable cellular automata thus obtained has a simple structure arising from permutations; the fundamental period of arbitrary configuration in the time evolution is exactly computed.

2. Reversibility of cellular automata

A one-dimensional cellular automaton $\mathcal{A}_r^{(n)} = \langle N, \mathbb{Z}_n, E, \delta\rangle$ is a quadruple defined by the one-dimensional array of N cells; a finite commutative ring $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$; a neighbourhood $E := \{e_1, e_2, \dots, e_r\}$, where $e_1, e_2, \dots, e_r$ are consecutive r integers; and a local transition function $\delta : \mathbb{Z}_n^r \to \mathbb{Z}_n$, where $\mathbb{Z}_n^r$ is the direct product of r copies of the ring $\mathbb{Z}$. In this paper, a cellular automaton is assumed to be with periodic boundaries. The cellular automaton $\mathcal{A}_r^{(n)}$ has n^{n^r} local transition functions, each of which is called a rule and is referred to the number

$$R := \sum_{c_1, c_2, \dots, c_r \in \mathbb{Z}_n} \delta(c_1, c_2, \dots, c_r) n^{n^{r-1}c_1 + n^{r-2}c_2 + \dots + n^0 c_r}, \tag{1}$$

where the sum ranges over possible combinations of $c_1, c_2, \dots, c_r \in \mathbb{Z}_n$. A mapping $c : \mathbb{Z}_N \to \mathbb{Z}_n$ is called a configuration of size N and is updated by the global transition function F:

$$F(c) = c',$$

where

$$c'_i = \delta\big(c_{i+e_1}, c_{i+e_2}, \dots, c_{i+e_r}\big),$$

for $i = 1, 2, \dots, N$ and $c_i \in \mathbb{Z}_n$ is the value of the ith cell in the configuration c. If the global transition F is bijective then the cellular automaton $\mathcal{A}_r^{(n)}$ is called reversible.

In order to show reversibility of cellular automata, graph-theoretical approaches are valid [15]. Each configuration of the cellular automaton $\mathcal{A}_r^{(n)}$ one-to-one corresponds to a closed walk of the de Bruijn graph $G_{r-1}^{(n)}$ [18]. Let $\mathcal{E}$ be the edge set of the de Bruijn graph $G_{r-1}^{(n)}$. Consider a mapping ϕ from $\mathcal{E}$ to $\mathbb{Z}_n$. Since $\mathcal{E}$ is equivalent to $\mathbb{Z}_n^r$ as a set, each mapping ϕ can be identified with a rule of $\mathcal{A}_r^{(n)}$. A mapping ϕ which is identified with the rule referred to the number R is denoted by ϕ_R. By the mapping $\phi_R : \mathcal{E} \to \mathbb{Z}_n$, each closed walk in $G_{r-1}^{(n)}$ corresponds to a configuration of $\mathcal{A}_r^{(n)}$ [19, 20]. If the correspondence is injective then the rule is reversible because the number of possible configurations of $\mathcal{A}_r^{(n)}$ is n^N and $G_{r-1}^{(n)}$ has n^N closed walks of length N for any $N > r - 1$.

Let us consider a weighted adjacency matrix $M_R G_{r-1}^{(n)}$ of the de Bruijn graph $G_{r-1}^{(n)}$ [21]. The matrix $M_R G_{r-1}^{(n)}$ is a $n^{r-1} \times n^{r-1}$ matrix whose entries m_{ij} $(i, j = 1, 2, \dots, n^{r-1})$ are given by

$$m_{ij} = \begin{cases} w_k & \text{if} \quad v_i v_j \text{ is an edge and } \phi_R(v_i v_j) = k \\ 0 & \text{otherwise,} \end{cases}$$

Table 1. All reversible rules of ECA with periodic boundaries and the configuration sizes for which the rules are reversible.

Rule		Size
150 105		$N \equiv 1, 2 \pmod 3$
154 166 180 210	45 75 89 101	$N \equiv 1 \pmod 2$
170 240 15 85		All $N \in \mathbb{N}$
204 51		All $N \in \mathbb{N}$

for $k = 0, 1, \ldots, n-1$, where v_i and v_j are vertices of $G_{r-1}^{(n)}$ and $v_i v_j$ denotes a directed edge which connects v_i and v_j in the direction from v_i to v_j. We regard the weighted adjacency matrix as a matrix over $T(W)$, where $T(W)$ is the tensor algebra $T(W) = \bigoplus_{i=0}^{\infty} W^{\otimes i}$ and W is the $\mathbb{Z}$-module generated by the weights $w_0, w_1, \ldots, w_{n-1}$. Then the entry in position (i, j) of the Nth power of the matrix $M_R G_{r-1}^{(n)}$ has the following form:

$$\sum_{k_1, k_2, \ldots, k_{N-1} \in \{1, 2, \ldots, n^{r-1}\}} m_{ik_1} \otimes m_{k_1 k_2} \otimes \cdots \otimes m_{k_{N-1} j}.$$

We obtain the following theorem [15].

Theorem 1. *A rule of the cellular automaton* $\mathcal{A}_r^{(n)} = \langle N, \mathbb{Z}_n, E, \delta \rangle$ *with periodic boundaries is reversible if and only if all* n^N *terms of* $\mathrm{tr}\big[\big(M_R G_{r-1}^{(n)}\big)^N\big]$ *are distinct.*

Let $n = 2$ and $r = 3$. Then we obtain ECA, $\mathcal{A}_3^{(2)} = \langle N, \mathbb{Z}_2, \{-1, 0, 1\}, \delta \rangle$. For ECA, we can inductively compute the trace of the Nth power of the weighted adjacency matrix $M_R G_2^{(2)}$, and obtain the following theorem concerning reversibility of ECA [15].

Theorem 2. *There exist exactly 16 reversible rules of ECA with periodic boundaries (see table 1).*

The rules in the last two rows of table 1 are trivial and those in the first row are additive, i.e., their updating rules are sum modulo two of the values of the neighbours [4]. Therefore, only the rules in the second row are considered to be nonlinear. In the next section, we show that the initial value problems for the rules 154, 166, 180 and 210 with periodic boundaries are reduced to initial-boundary value problems for some linear systems on $\mathbb{Z}_2$.

3. Reduction to linear systems

Since the rules 154, 166, 180 and 210 are congruent with respect to the reflection and the conjugation [1], we consider only the rule 154. The size N of a configuration is assumed to be odd in order for the rule 154 to be reversible.

Let δ be the local transition of the rule 154. Also let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be the right shift,

$$f : c_i \mapsto c_{i+1},$$

for $i = 1, 2, \ldots, N$, where c is a configuration of odd size N. Consider the composition $\tilde{\delta} := f \circ \delta$. Then we obtain a reversible global transition $\tilde{F}$:

$$\tilde{F}(c) = c', \tag{2}$$

where

$$
\begin{aligned}
c_i' &= \tilde{\delta}\,(c_{i-2}, c_{i-1}, c_i) \\
&\equiv \begin{cases} c_{i-2} + c_i \pmod 2 & \text{if} \quad c_{i-1} = 0 \\ c_i \pmod 2 & \text{otherwise,} \end{cases}
\end{aligned}
\tag{3}
$$

for $i = 1, 2, \ldots, N$. Thus the value 1 (resp. 0) of a cell evolves into the value 0 (resp. 1) only when 10 hits it from the left. Therefore, a configuration is divided into some evolving domains by boundary walls, each of which consists of a sequence of 1's and is never updated by $\tilde{F}$.

3.1. Construction of blocks

The evolving domains of a configuration are extracted as follows.

(i) Draw lines under all 10's in the configuration.
(ii) Pick arbitrary one pair of the lined two adjacent cells whose next cell on the right is unlined.
(iii) If the next two cells on the right of the lined two adjacent cells are
  (a) 00 then draw a line under the 00 and go to (iii),
  (b) otherwise, draw a line under the left of the two adjacent cells and go to (ii).
(iv) Repeat the procedure (ii) as many as possible.

If we complete the above procedure then the right side of a pair of lined two adjacent cells is other pair of lined two adjacent cells or a lined cell. Therefore, by the above procedure, we obtain some blocks of lined adjacent cells, each of which consists of some pairs of lined two cells and a lined cell on the rightmost. We call each of such blocks of lined cells simply a block. We assume every lined cell to be in a block. By construction, the remaining (unlined) cells must take the value 1 and are never updated by the global transition $\tilde{F}$. In each block, the cells on the even positions counting from the left are never updated by $\tilde{F}$ as well. Since a block ends only when a cell on an even position takes the value 1, the block size is a conserved quantity.

Example 1. Let us consider a configuration

$$1\,0\,0\,1\,1\,1\,0\,1\,0\,1\,0\,0\,0\,0\,1\,0\,0\,1\,1\,0\,1\,0\,1\,1\,1.$$

At first, draw lines under all 10's

$$\underline{1\,0}\,0\,1\,1\,\underline{1\,0}\,\underline{1\,0}\,\underline{1\,0}\,0\,0\,0\,\underline{1\,0}\,0\,1\,\underline{1\,0}\,\underline{1\,0}\,1\,1\,1.$$

Pick, for example, the leftmost lined two adjacent cells then, by (iii), we have

$$\underline{1\,0}\,\underline{0}\,1\,1\,\underline{1\,0}\,\underline{1\,0}\,\underline{1\,0}\,0\,0\,0\,\underline{1\,0}\,0\,1\,\underline{1\,0}\,\underline{1\,0}\,1\,1\,1.$$

Repeating the procedure, we finally obtain the following four blocks of size 3, 9, 3 and 5, respectively,

$$\underbrace{\underline{1\,0}\,\underline{0}}_{\text{size}\,3}\;1\,1\;\underbrace{\underline{1\,0}\,\underline{1\,0}\,\underline{1\,0}\,\underline{0\,0}\,\underline{0}}_{\text{size}\,9}\;\underbrace{\underline{1\,0}\,\underline{0}}_{\text{size}\,3}\;1\;\underbrace{\underline{1\,0}\,\underline{1\,0}\,\underline{1}}_{\text{size}\,5}\;1\,1.$$

3.2. Time evolution of blocks

Now we consider the time evolution of blocks given by the global transition $\tilde{F}$. Suppose the values of the cells in a block of size $2L+1$ at time t to be $b_0^t, b_1^t, \ldots, b_{2L}^t$, where L is a natural number. By construction, the values of the cells $b_0^t, b_1^t, \ldots, b_{2L}^t$ naturally satisfy

$$b_0^t = 1 \qquad \text{and} \qquad b_{2i+1}^t = 0, \tag{4}$$

for $i = 0, 1, \ldots, L-1$ and $t \geqslant 0$. Hence, the initial condition for the block is

$$b_0^0 = 1 \qquad \text{and} \qquad b_{2i+1}^0 = 0, \tag{5}$$

for $i = 0, 1, \ldots, L-1$. According to (4) and the local transition (3), the values of the cells satisfy the following boundary conditions:

$$b_0^t = 1 \qquad \text{and} \qquad b_1^t = 0, \tag{6}$$

for $t \geqslant 0$.

There exist eight possible configurations of the neighbourhood of a cell. However, according to the condition (4), only five configurations 101, 100, 010, 001 and 000 of them can be realized in a block. Therefore, we need only five local transitions of the rule 154 in the time evolution of a block:

$$\delta(1,0,0) = \delta(0,0,1) = 1$$

$$\delta(1,0,1) = \delta(0,1,0) = \delta(0,0,0) = 0.$$

Hence the time evolution of the block is represented by

$$b_i^{t+1} \equiv b_{i-2}^t + b_i^t \pmod 2, \tag{7}$$

for $i = 2, 3, \ldots, 2L$. This is nothing but a linear system on $\mathbb{Z}_2$. Thus the initial value problem for the rule 154 with periodic boundaries is reduced to the initial-boundary value problem (5), (6) for the linear system (7) on $\mathbb{Z}_2$.

Each configuration of a cellular automaton is represented by a generating function [4]. Noting the condition (4), let us consider the following generating function for a configuration of the block

$$B(x)^t := 1 + \sum_{i=1}^{L} b_{2i}^t x^{2i},$$

where the value of the ith cell is the coefficient of x^i, and all coefficients are elements of the ring $\mathbb{Z}_2$. By (7), the time evolution of the block is represented by multiplication of the generating function for the configuration by the polynomial

$$T(x) := 1 + x^2,$$

according to

$$B(x)^{t+1} = T(x)B(x)^t,$$

where the coefficient of x^i for $i > 2L$ is assumed to be zero.

The inverse time evolution of the block is also represented by a polynomial

$$T^{-1}(x) = \sum_{i=0}^{L} x^{2i}.$$

Therefore, we have

$$\begin{aligned} B(x)^{t-1} &= T(x)^{-1} B(x)^t \\ &= 1 + \sum_{i=1}^{L} \left(1 + \sum_{j=1}^{i} b_{2j}^t \right) x^{2i}. \end{aligned}$$

Thus the value of a cell in a block at the previous time step is given by the sum modulo two of the values of its own and all cells on the left side at the present time step.

3.3. Periods

Let

$$s := \lfloor \log_2 L \rfloor, \tag{8}$$

where $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function. Then 2^{s+1} is the smallest number satisfying

$$B(x)^{t+2^{s+1}} = T(x)^{2^{s+1}} B(x)^t = B(x)^t$$

for any $B(x)^t$ (see proposition 2). The smallest $P \in \mathbb{N}$ satisfying $B(x)^{t+P} = B(x)^t$ is called the fundamental period of the configuration $B(x)^t$ in the time evolution $T(x)$. Thus the fundamental period of a block whose size is $2L + 1$ is 2^{s+1}. Note that the fundamental period of a block depends not on its configuration but only on its size.

Consider a configuration c of size N which contains m blocks of size $2L_1 + 1, 2L_2 + 1, \ldots, 2L_m + 1$, where $L_1 \leqslant L_2 \leqslant \cdots \leqslant L_m \in \mathbb{N}$, respectively. Let

$$s^i := \lfloor \log_2 L_i \rfloor,$$

for $i = 1, 2, \ldots, m$. Then the fundamental period of the configuration c in the time evolution given by the global transition $\tilde{F}$ is the least common multiple of $2^{s^1+1}, 2^{s^2+1}, \ldots, 2^{s^m+1}$:

$$\mathrm{lcm}\left(2^{s^1+1}, 2^{s^2+1}, \ldots, 2^{s^m+1}\right) = 2^{s^m+1}.$$

The global transition $\tilde{F}$ is given by the composition $\tilde{\delta} = f \circ \delta$ of the local transition δ of the rule 154 and the right shift f. Now we consider the global transition F which is given only by the local transition δ of the rule 154. Then the configuration which is obtained after 2^{s^m+1} step evolution of the configuration c by the global transition F is the 2^{s^m+1} shift of c in the left direction. The size N of the configuration c and the number 2^{s^m+1} are relatively prime, because N is odd. Thus we have the following proposition.

Proposition 1. *Let c be a configuration of size $N \not\equiv 0 \pmod 2$ which contains m blocks of size $2L_1+1, 2L_2+1, \ldots, 2L_m+1$ $(L_1 \leqslant L_2 \leqslant \cdots \leqslant L_m \in \mathbb{N})$, respectively. Then the fundamental period of the configuration c in the rule 154 is a divisor of $2^{s^m+1}N$. If the configuration c has no translation symmetry then the fundamental period is exactly $2^{s^m+1}N$.*

Since the fundamental period of a block depends only on its size, the maximum among the fundamental periods for a fixed N in the rule 154 is attained by the configurations which consist only of a block.

Corollary 1. *The maximum $P_{\max}$ among the fundamental periods of configurations for a fixed size N in the rule 154 satisfies*

$$N(N+1) \leqslant 2P_{\max} \leqslant 2N(N-1).$$

4. Generalizations

In the previous section, we showed that the initial value problem for the reversible rule 154 of ECA with periodic boundaries can be reduced to the initial-boundary value problem (5), (6) for the linear system (7). The following properties of the reversible ECA characterize its behaviour in the time evolution given by the global transition $\tilde{F}$.

(i) In every configuration, the time evolution is localized in some blocks which are separated by boundary walls.
(ii) In each block, the time evolution is linear on a finite commutative ring.

We generalize the linearizable ECA to the cellular automaton $\mathcal{A}_r^{(n)}$ for arbitrary $n \in \mathbb{N}$ and $r \in \mathbb{N}$ with the above properties (i) and (ii) kept.

4.1. Rules

Remember that, in the time evolution given by the global transition $\tilde{F}(2)$ of the right-shifted rule 154, the value 1 (resp. 0) of a cell evolves into the value 0 (resp. 1) only when 10 hits it from the left. Thus a sequence of numbers 10 acts as a permutation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on the finite commutative ring $\mathbb{Z}_2$. Noting this, we consider the following cellular automaton; a sequence of numbers $i0\cdots0$ $(i = 0, 1, \ldots, n-1)$ of size $r-1$ acts as a permutation

$$\begin{pmatrix} 0 & 1 & \cdots & n-1 \\ \alpha i & \alpha i + 1 & \cdots & \alpha i + n - 1 \end{pmatrix} \tag{9}$$

on the finite commutative ring $\mathbb{Z}_n$, where $0 \leqslant \alpha \leqslant n-1$ and the arithmetic is considered modulo n. The value $c_j \in \mathbb{Z}_n$ $(j = 1, 2, \ldots, N)$ of the jth cell in a configuration c evolves into $\alpha i + c_j \pmod n$ only when $i0\cdots0$ $(i = 0, 1, \ldots, n-1)$ hits it from the left. Thus we obtain the local transition $\tilde{\delta} : \mathbb{Z}_n^r \to \mathbb{Z}_n$:

$$\begin{aligned} c_i' &= \tilde{\delta}\left(c_{i+e_1-e_r}, c_{i+e_2-e_r}, \ldots, c_i\right) \\ &\equiv \begin{cases} \alpha c_{i+e_1-e_r} + c_i \pmod n & \text{if} \quad c_{i+e_2-e_r} = \cdots = c_{i+e_{r-1}-e_r} = 0 \\ c_i \pmod n & \text{otherwise,} \end{cases} \end{aligned} \tag{10}$$

for $i = 1, 2, \ldots, N$. Remark that the local transition $\tilde{\delta}$ is the composition $f \circ \delta$ of the right shift $f : \mathbb{Z}_n \to \mathbb{Z}_n$:

$$f : c_i \mapsto c_{i+e_r},$$

for $i = 1, 2, \ldots, N$ and a local transition $\delta : \mathbb{Z}_n^r \to \mathbb{Z}_n$ of the cellular automaton $\mathcal{A}_r^{(n)}$. If $\alpha = 0$ then the rule (10) is nothing but the identity $c_i' = c_i$ for $i = 1, 2, \ldots, N$.

Consider the global transition $\tilde{F}$:

$$\tilde{F}(c) = c',$$

where c' is given by the local transition (10). Then $\tilde{F}$ actually has the properties (i) and (ii); we can extract blocks from a configuration as follows.

(i) Draw lines under all $i0\cdots0$'s $(i = 1, 2, \ldots, n-1)$ of size $r-1$ in the configuration.
(ii) Pick arbitrary one pair of the lined $r-1$ adjacent cells whose next cell on the right is unlined.
(iii) If the next $r-1$ cells on the right of the lined $r-1$ adjacent cells are
   (a) $0\cdots0$ then draw a line under the $0\cdots0$ and go to (iii),

(b) otherwise, draw a line under the leftmost of the $r-1$ adjacent cells and go to (ii).

(iv) Repeat the procedure (ii) as many as possible.

If the size of a configuration is not equal to a multiple of $r-1$ then this procedure ends in finite steps, and we obtain some blocks of size 1 (mod $r-1$), each of which consists of some pairs of lined $r-1$ adjacent cells and a lined cell on the rightmost. We assume every lined cell to be in a block. In each block, only the cells in position $i \equiv 1 \pmod{r-1}$ are updated by the global transition $\tilde{F}$. The remaining (unlined) cells are never updated by $\tilde{F}$. Since the sequence of numbers $0\cdots 0$ acts as the identity, the leftmost cell of each block is never updated by $\tilde{F}$ as well. Therefore, the size of a block is a conserved quantity. Since the time evolution of a block is given by a linear equation on $\mathbb{Z}_n$ as we mention later, the rule (10) is reversible if the size of a configuration is not equal to a multiple of $r-1$.

Remark 1. Let the ith vertex v_i of the de Bruijn graph $G_{r-1}^{(n)}$ associated with the cellular automaton $\mathcal{A}_r^{(n)}$ be $v_i = (i-1)_n$, where the subscript n means the n-adic number expression[3]. Consider the weighted adjacency matrix $M_R G_{r-1}^{(n)}$ of the graph $G_{r-1}^{(n)}$ whose entries are weighted according to the rule (10). Put the sum of all n^{r-1} elements in the ith row of the matrix S_i for $i = 1, 2, \ldots, n^{r-1}$. The weighted adjacency matrix $M_R G_{r-1}^{(n)}$ of the de Bruijn graph $G_{r-1}^{(n)}$ has the following property: every row sum S_i equals to both the trace and the weight sum $\sum_{j=0}^{n-1} w_j$,

$$S_i = \mathrm{tr}\big[M_R G_{r-1}^{(n)}\big] = \sum_{j=0}^{n-1} w_j \tag{11}$$

for $i = 1, 2, \ldots, n^{r-1}$. Because the matrix $M_R G_{r-1}^{(n)}$ is obtained by an action of the permutation (9) on the rows of the weighted adjacency matrix whose entries are weighted according to the rule equivalent to the left shift $c_i' = c_{i+e_r}$, and the action never changes the trace and all S_i's. By the setting of the vertices v_i, it is clear that the weighted adjacency matrix associated with the left shift satisfies the condition (11). As we mentioned in [15], this property with $n = 2$ and $r = 3$ is a sufficient condition for reversibility of ECA.

4.2. Time evolution

Suppose the values of the cells in a block of size $(r-1)L+1$ to be $b_0^t, b_1^t, \ldots, b_{(r-1)L}^t$, where L is a natural number. Then the time evolution of the block given by the global transition $\tilde{F}$ is represented by a linear equation on the finite commutative ring $\mathbb{Z}_n$

$$b_i^{t+1} \equiv \alpha b_{i-r+1}^t + b_i^t \pmod{n} \tag{12}$$

for $i = r-1, r, \ldots, (r-1)L$.

By construction of the block, the values $b_0^t, b_1^t, \ldots, b_{(r-1)L}^t$ of the cells in the block naturally satisfy

$$b_0^t = b_0^0 \neq 0 \qquad \text{and} \qquad b_j^t = 0, \tag{13}$$

for $j \neq r-1, 2(r-1), \ldots, (r-1)L$ and $t \geqslant 0$. Therefore, the initial condition for the block is

$$b_0^0 \neq 0 \qquad \text{and} \qquad b_j^0 = 0, \tag{14}$$

for $j \neq r-1, 2(r-1), \ldots, (r-1)L$. The block also satisfies the following boundary conditions:

$$b_0^t = b_0^0 \qquad \text{and} \qquad b_1^t = b_2^t = \cdots = b_{r-2}^t = 0, \tag{15}$$

[3] If the length of the n-adic number is shorter than $r-1$ then we fill 0's on the left side so the length as to be $r-1$.

for $t \geqslant 0$. Thus the initial value problem for the rule (10) with periodic boundaries is reduced to the initial-boundary value problem (14), (15) for the linear system (12) on the finite commutative ring $\mathbb{Z}_n$.

By virtue of the condition (13), the generating function for a configuration of the block is given as follows:

$$B(x)^t := b_0^0 + \sum_{i=1}^{L} b_{(r-1)i}^t x^{(r-1)i},$$

where the coefficients are elements of the ring $\mathbb{Z}_n$. By (12), the time evolution of the block is represented by multiplication of the generating function for the configuration by the polynomial

$$T(x) := 1 + \alpha x^{r-1},$$

where the coefficient of x^i for $i > (r-1)L$ is assumed to be zero.

4.3. Periods

Now we consider the fundamental period of a block in the rule (10). The time evolution of a block whose generating function is $B(x)^t$ is given by the polynomial $T(x)$ on $\mathbb{Z}_n$. Therefore, computing a period of the block is equivalent to finding such a natural number M that $T(x)^M = 1$ on $\mathbb{Z}_n$. Since the coefficient of x^i for $i > (r-1)L$ in the polynomial $T(x)^M$ is assumed to be zero, $T(x)^M$ is expanded as follows:

$$T(x)^M = \sum_{j=0}^{L} {}_M C_j \alpha^j x^{(r-1)j}. \tag{16}$$

Thus, in order to compute a period of a block of size L in the rule (10), we find such $M \in \mathbb{N}$ that ${}_M C_j \alpha^j \equiv 0 \pmod{n}$ for $1 \leqslant j \leqslant L$. Remark that this condition for the binomial coefficients of the polynomial $T(x)^M$ is independent of the parameter r in $T(x)$, which is the size of the neighbourhood of the cellular automaton $\mathcal{A}_r^{(n)}$.

Now, let $n = \prod_{i=1}^{q} p_i^{e_{p_i}(n)}$, where $p_1 < p_2 < \cdots < p_q$ are primes and $e_{p_i}(n)$ is the index of n with respect to p_i for $i = 1, 2, \ldots, q$. Also let $\alpha = \beta \prod_{i=1}^{q} p_i^{e_{p_i}(\alpha)} \neq 0$, where β and n are relatively prime. Put

$$s_i := \lfloor \log_{p_i} L \rfloor$$

for $i = 1, 2, \ldots, q$. Also put

$$M := \prod_{i=1}^{q} p_i^{e_{p_i}(M)},$$

where

$$e_{p_i}(M) = \begin{cases} e_{p_i}(n) + s_i & \text{if} \quad i \in I_0 \\ e_{p_i}(n) - e_{p_i}(\alpha) & \text{if} \quad i \in I_1 \\ 0 & \text{if} \quad i \in I_2 \end{cases}$$

and

$$I_0 := \{i \in \{1, 2, \ldots, q\} | e_{p_i}(\alpha) = 0\}$$
$$I_1 := \{i \in \{1, 2, \ldots, q\} | 1 \leqslant e_{p_i}(\alpha) < e_{p_i}(n)\}$$
$$I_2 := \{i \in \{1, 2, \ldots, q\} | e_{p_i}(n) \leqslant e_{p_i}(\alpha)\}.$$

Then we have the following lemma concerning the binomial coefficients.

Lemma 1.

(i) If $1 \leqslant j \leqslant L$ *then*

$$_MC_j\alpha^j \equiv 0 \pmod n.$$

(ii) Let $M' < M$ *be a divisor of* M*. Then there exists* $1 \leqslant j \leqslant L$ *satisfying*

$$_{M'}C_j\alpha^j \not\equiv 0 \pmod n.$$

Proof.

(i) Since $s_i = \lfloor \log_{p_i} L \rfloor$, $L < p_i^{s_i+1}$ holds for any $i = 1, 2, \ldots, q$. Therefore, we show that if $1 \leqslant j < p_i^{s_i+1}$ then $_MC_j\alpha^j \equiv 0 \pmod{p_i^{e_{p_i}(n)}}$ for $i = 1, 2, \ldots, q$. Then, by the Chinese remainder theorem [22], we have $_MC_j\alpha^j \equiv 0 \pmod n$ for $1 \leqslant j \leqslant L$.

For $i \in I_2$, because $\alpha \equiv 0 \pmod{p_i^{e_{p_i}(n)}}$ holds, it follows $_MC_j\alpha^j \equiv 0 \pmod{p_i^{e_{p_i}(n)}}$ for $1 \leqslant j < p_i^{s_i+1}$. Suppose $i \in I_1$. Then the index $e_{p_i}(_MC_1\alpha)$ for $j = 1$ is $e_{p_i}(n)$:

$$e_{p_i}(_MC_1\alpha) = e_{p_i}(n) - e_{p_i}(\alpha) + e_{p_i}(\alpha) = e_{p_i}(n).$$

Since $1 \leqslant e_{p_i}(\alpha)$, the index $e_{p_i}(_MC_j\alpha^j)$ is monotone increasing with respect to j. Therefore, we have $e_{p_i}(_MC_j\alpha^j) \geqslant e_{p_i}(n)$ and hence $_MC_j\alpha^j \equiv 0 \pmod{p_i^{e_{p_i}(n)}}$ for $1 \leqslant j < p_i^{s_i+1}$.

Finally, suppose $i \in I_0$. Since $e_{p_i}(\alpha) = 0$, the index $e_{p_i}(_MC_j\alpha^j)$ is invariable with respect to j except such j that $p_i^u | j$ for $1 \leqslant u \leqslant s_i$. Actually, for $j = 1$, we have

$$e_{p_i}(_MC_1\alpha) = e_{p_i}(M) = e_{p_i}(n) + s_i.$$

The index $e_{p_i}(_MC_j\alpha^j)$ remains invariable for $1 \leqslant j \leqslant p_i - 1$ because of

$$e_{p_i}(M-1) = e_{p_i}(M-2) = \cdots = e_{p_i}(M - p_i + 1) = 0$$

and

$$e_{p_i}(p_i - 1) = e_{p_i}(p_i - 2) = \cdots = e_{p_i}(1) = 0.$$

For $j = p_i$, the prime p_i appears in the denominator of the binomial coefficient $_MC_{p_i}$:

$$_MC_{p_i} = \frac{M(M-1)\cdots\{M-(p_i-1)\}}{1 \times 2 \times \cdots \times p_i}.$$

Hence we have

$$e_{p_i}\left(_MC_{p_i}\alpha^{p_i}\right) = e_{p_i}(n) + s_i - 1.$$

For $j = p_i + 1$, however, p_i appears in the numerator of $_MC_{p_i+1}$ again

$$\begin{aligned} _MC_{p_i+1} &= {_MC_{p_i}}\frac{M - p_i}{p_i + 1} \\ &= {_MC_{p_i}}\frac{p_i(M/p_i - 1)}{p_i + 1}, \end{aligned}$$

therefore the index recovers its initial value:

$$e_{p_i}\left(_MC_{p_i+1}\alpha^{p_i+1}\right) = e_{p_i}(n) + s_i.$$

Thus, for $1 \leqslant j < p^{s_i+1}$, we inductively have

$$e_{p_i}(_MC_j\alpha^j) = \begin{cases} e_{p_i}(n) + s_i - u & \text{if } \; p_i^u | j \text{ for } 1 \leqslant u \leqslant s_i \\ e_{p_i}(n) + s_i & \text{otherwise.} \end{cases}$$

Noting $1 \leqslant u \leqslant s_i$, we have $e_{p_i}(_MC_j\alpha^j) \geqslant e_{p_i}(n)$ and hence $_MC_j\alpha^j \equiv 0 \pmod{p_i^{e_{p_i}(n)}}$ for $1 \leqslant j < p_i^{s_i+1}$. The Chinese remainder theorem completes the proof.

(ii) Note that, for $i \in I_2$, M/p_i is not a divisor of M because p_i does not divide M. Consider a divisor $M' = M/p_i$ of M. If $i \in I_0$ then we have

$$e_{p_i}\left({}_{M'}C_{p_i^{s_i}}\alpha^{p_i^{s_i}}\right) = e_{p_i}(n) - 1,$$

therefore ${}_{M'}C_{p_i^{s_i}}\alpha^{p_i^{s_i}} \not\equiv 0 \pmod n$. Since $s_i = \lfloor \log_{p_i} L \rfloor$, $j = p_i^{s_i}$ satisfies $1 \leqslant j \leqslant L$.

On the other hand, if $i \in I_1$ then the index $e_{p_i}({}_{M'}C_j\alpha^j)$ is monotone increasing with respect to j. Therefore the minimum is attained at $j = 1$,

$$e_{p_i}({}_{M'}C_1\alpha) = e_{p_i}(n) - 1.$$

Thus we have ${}_{M'}C_1\alpha \not\equiv 0 \pmod n$.

For smaller divisors M' of M, we inductively have ${}_{M'}C_j\alpha^j \not\equiv 0 \pmod n$ for $1 \leqslant {}^{\exists}j \leqslant L$. □

Noting expansion (16) of the polynomial $T(x)^M$ on $\mathbb{Z}_n$, lemma 1(i) leads to

$$T(x)^M = 1.$$

Thus M is a period of the block of size L in the rule (10).

By lemma 1(ii), M is the smallest number satisfying $T(x)^M = 1$. If the commutative ring $\mathbb{Z}_n$ has zero divisors then there can be $M' < M$ which satisfy $T(x)^{M'}B(x)^0 = B(x)^0$ for an initial configuration $B(x)^0$ of the block. The smallest one among such M''s must be a divisor of M. Therefore, the fundamental period of the block is a divisor of M, which depends on the initial configuration $B(x)^0$ of the block.

We obtain the following proposition concerning the fundamental period of a block in the rule (10).

Proposition 2. *Let*

$$\eta_i(k) := \min[s_i, k-1],$$

$$\xi_i(k) := \min_{d=0}^{\eta_i(k)}\left[\min_{j=0}^{L-p_i^{s_i-d}}\left[\gcd\left(b^0_{(r-1)j}p_i^d,\, p_i^k\right)\right]\right]$$

and

$$\varpi_i(k) := \begin{cases} p_i^k & \text{if} \quad \xi_i(k) = p_i^k \\ 1 & \text{otherwise,}\end{cases}$$

for $i = 1, 2, \ldots, q$ *and* $k = 1, 2, \ldots, e_{p_i}(n) + s_i$. *Put*

$$Q_0 := \prod_{i \in I_0} \max_{k=1}^{e_{p_i}(n)+s_i}[\varpi_i(k)]$$

$$Q_1 := \prod_{i \in I_1} \min_{j=0}^{L-1}\left[\gcd\left(b^0_{(r-1)j},\, p_i^{e_{p_i}(n)-e_{p_i}(\alpha)}\right)\right],$$

where $b^0_0, b^0_{(r-1)}, \ldots, b^0_{(r-1)L}$ *are the initial values of the cells in a block of size* L. *Then the fundamental period of the block in the rule (10) is*

$$\frac{M}{Q_0 Q_1}.$$

Proof. Suppose $i \in I_0$. Consider a divisor $M' = M\big/p_i^k$ of M, where $1 \leqslant k \leqslant e_{p_i}(n) + s_i$. If and only if $p_i^{s_i-\eta_i(k)}\big| j$ we have ${}_{M'}C_j\alpha^j \not\equiv 0 \pmod n$ because of

$$e_{p_i}\left({}_{M'}C_{p_i^{s_i-\eta_i(k)}}\alpha^{p_i^{s_i-\eta_i(k)}}\right) = e_{p_i}(n) - k + \eta_i(k) < e_{p_i}(n).$$

Note that if $k > s_i$ then $p^{s_i-\eta_i(k)} = 1$ and hence ${}_{M'}C_j\alpha^j \not\equiv 0 \pmod n$ for $1 \leqslant {}^{\forall}j \leqslant L$.

Consider the polynomial $T(x)^{M'}B(x)^0$ in x on $\mathbb{Z}_n$. If the coefficient ${}_{M'}C_l\alpha^l b^0_{(r-1)j}$ of $x^{(r-1)(l+j)}$ does not vanish then $p_i^{s_i-\eta_i(k)}\big| l$. The coefficient of $x^{(r-1)m}$ in $T(x)^{M'}B(x)^0$ therefore vanish for $m < p_i^{s_i-\eta_i(k)}$. Assume $l = p_i^{s_i-d}$ for $0 \leqslant {}^{\exists}d \leqslant \eta_i(k)$. If $e_{p_i}\left(b^0_{(r-1)j}\right) \geqslant k-d$ holds for $0 \leqslant j \leqslant L-l$ then we have

$$\begin{aligned} e_{p_i}\left({}_{M'}C_l\alpha^l b^0_{(r-1)j}\right) &= e_{p_i}({}_{M'}C_l\alpha^l) + e_{p_i}\left(b^0_{(r-1)j}\right) \\ &\geqslant \left(e_{p_i}(n) - k + d\right) + (k-d) = e_{p_i}(n) \end{aligned}$$

and hence ${}_{M'}C_l\alpha^l b^0_{(r-1)j} \equiv 0 \pmod n$. This implies that the coefficient of $x^{(r-1)m}$ in $T(x)^{M'}B(x)^0$ vanish for $l = p_i^{s_i-d} \leqslant m \leqslant L$. Thus, if we have

$$\min_{j=0}^{L-p_i^{s_i-d}}\left[\gcd\left(b^0_{(r-1)j}, p_i^{k-d}\right)\right] = p_i^{k-d}$$

for $0 \leqslant {}^{\forall}d \leqslant \eta_i(k)$, or equivalently have

$$\xi_i(k) = \min_{d=0}^{\eta_i(k)}\left[\min_{j=0}^{L-p_i^{s_i-d}}\left[\gcd\left(b^0_{(r-1)j}p_i^d, p_i^k\right)\right]\right] = p_i^k$$

then $T(x)^{M'}B(x)^0 = B(x)^0$ holds on $\mathbb{Z}_n$. Therefore, the divisor $M' = M\big/p_i^k$ of M is a period of the block in the rule (10). The maximum $p_i^k \in \{1, p_i, \ldots, p_i^{e_{p_i}(n)+s_i}\}$ for which $M' = M/p_i^k$ is a period of the block in the rule (10) is

$$\max_{k=1}^{e_{p_i}(n)+s_i}[\varpi_i(k)].$$

On the other hand, suppose $i \in I_1$. Consider a divisor $M' = M\big/p_i^k$ of M, where $1 \leqslant k \leqslant e_{p_i}(n) - e_{p_i}(\alpha)$. The smallest j for which ${}_{M'}C_j\alpha^j \not\equiv 0 \left(\mathrm{mod}\ p_i^{e_{p_i}(n)}\right)$ is $j = 1$,

$$e_{p_i}({}_{M'}C_1\alpha) = e_{p_i}(n) - k.$$

Since $e_{p_i}({}_{M'}C_j\alpha^j)$ is monotone increasing with respect to j, if we have

$$\min_{j=0}^{L-1}\left[\gcd\left(b^0_{(r-1)j}, p_i^k\right)\right] = p_i^k$$

then $T(x)^{M'}B(x)^0 = B(x)^0$ holds on $\mathbb{Z}_n$. Therefore the divisor $M' = M\big/p_i^k$ of M is a period of the block in the rule (10). The maximum $p_i^k \in \left\{1, p_i, \ldots, p_i^{e_{p_i}(n)-e_{p_i}(\alpha)}\right\}$ for which $M' = M\big/p_i^k$ is a period of the block in the rule (10) is

$$\min_{j=0}^{L-1}\left[\gcd\left(b^0_{(r-1)j}, p_i^{e_{p_i}(n)-e_{p_i}(\alpha)}\right)\right].$$

Thus we inductively obtain the desired result. □

Remark 2. If $e_{p_i}(\alpha) > 0$ holds for all $i = 1, 2, \ldots, q$, then we have

$$e_{p_i}(M) = \max\left[e_{p_i}(n) - e_{p_i}(\alpha), 0\right]$$

for all $i = 1, 2, \ldots, q$. This implies that the fundamental period of a block in the rule (10) with such α as above is a constant which is independent of its size.

Consider a configuration c of size $N \not\equiv 0 \pmod{r-1}$ which contains m blocks of size $(r-1)L_1 + 1, (r-1)L_2 + 1, \ldots, (r-1)L_m + 1$ ($L_1 \leqslant L_2 \leqslant \cdots \leqslant L_m \in \mathbb{N}$), respectively. Suppose the values of the cells in the block of size $(r-1)L_l + 1$ to be $b^t_{l,0}, b^t_{l,1}, \ldots, b^t_{l,(r-1)L_l}$ for $l = 1, 2, \ldots, m$. Let

$$s_i^l := \left\lfloor \log_{p_i} L_l \right\rfloor$$

for $i = 1, 2, \ldots, q$ and $l = 1, 2, \ldots, m$. Also let

$$\eta_i^l(k) := \min\left[s_i^l, k-1\right]$$

$$\xi_i^l(k) := \min_{d=0}^{\eta_i^l(k)}\left[\min_{j=0}^{L-p_i^{s_i^l-d}}\left[\gcd\left(b_{l,(r-1)j}^0 p_i^d, p_i^k\right)\right]\right]$$

$$\varpi_i^l(k) := \begin{cases} p_i^k & \text{if} \quad \xi_i^l(k) = p_i^k \\ 1 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots, q$, $l = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, e_{p_i}(n) + s_i^l$. Put

$$M^l := \prod_{i=1}^{q} p_i^{e_{p_i}(M^l)},$$

where

$$e_{p_i}(M^l) = \begin{cases} e_{p_i}(n) + s_i^l & \text{if} \quad i \in I_0 \\ e_{p_i}(n) - e_{p_i}(\alpha) & \text{if} \quad i \in I_1 \\ 0 & \text{if} \quad i \in I_2. \end{cases}$$

Also put

$$Q_0^l := \prod_{i \in I_0} \max_{k=1}^{e_{p_i}(n)+s_i^l}\left[\varpi_i^l(k)\right]$$

$$Q_1^l := \prod_{i \in I_1} \min_{j=0}^{L_l-1}\left[\gcd\left(b_{l,(r-1)j}^0, p_i^{e_{p_i}(n)-e_{p_i}(\alpha)}\right)\right]$$

for $l = 1, 2, \ldots, m$. Then, by proposition 2, the least common multiple

$$P := \operatorname{lcm}\left(\frac{M^1}{Q_0^1 Q_1^1}, \frac{M^2}{Q_0^2 Q_1^2}, \ldots, \frac{M^m}{Q_0^m Q_1^m}\right)$$

of the fundamental periods of the blocks is the fundamental period of the configuration c in the rule (10).

Now we consider the global transition F which depends not on the right shift f but only on the local transition δ of a rule of the cellular automaton $\mathcal{A}_r^{(n)}$,

$$F(c) = c',$$

where

$$\begin{aligned} c_i' &= \delta\left(c_{i+e_1}, c_{i+e_2}, \ldots, c_{i+e_r}\right) \\ &\equiv \begin{cases} \alpha c_{i+e_1} + c_{i+e_r} \pmod n & \text{if} \quad c_{i+e_2} = \cdots = c_{i+e_{r-1}} = 0 \\ c_{i+e_r} \pmod n & \text{otherwise,} \end{cases} \end{aligned} \tag{17}$$

for $i = 1, 2, \ldots, N$. Since P is the fundamental period of the configuration c in the time evolution given by the global transition $\tilde{F}$, the configuration which is obtained after P-step evolution of c by the global transition F is the $e_r P$ shift of c in the left direction. Thus we obtain the following proposition.

Proposition 3. *Let c be a configuration of size $N \not\equiv 0 \pmod{r-1}$ which contains m blocks of size $(r-1)L_1+1, (r-1)L_2+1, \ldots, (r-1)L_m+1$ ($L_1 \leqslant L_2 \leqslant \cdots \leqslant L_m \in \mathbb{N}$), respectively. Then the fundamental period of the configuration c in the rule (17) is a divisor of*

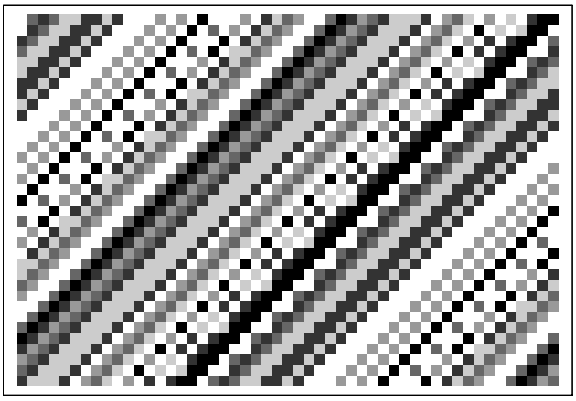

Figure 1. A greyscale image of the time evolution of the initial configuration of size 51 in the rule (19). Configurations at successive time steps are shown as successive lines from top to bottom. The values of the cells in the initial configuration are chosen randomly from 0 to 5. Cells with the highest value 5 are shown by black and with the lowest value 0 by white.

$$\frac{NP}{\gcd(N, e_r P)}. \tag{18}$$

If the configuration c has no translation symmetry then the fundamental period is exactly (18).

Example 2. Let $n = 6$ and $r = 3$. Also let $\alpha = 3$. Then the updating rule (17) is

$$c_i' \equiv \begin{cases} 3c_{i-1} + c_{i+1} \pmod 6 & \text{if} \quad c_i = 0 \\ c_{i+1} \pmod 6 & \text{otherwise}, \end{cases} \tag{19}$$

for $i = 1, 2, \ldots, N$. The rule number (1) is a number of 169 figures. A configuration is reversible in the rule (19) if its size is odd. An example of the time evolution is described in figure 1.

The initial configuration in figure 1 contains five blocks of size 3, 3, 3, 7 and 15 respectively. Hence we have

$$L_1 = 1, \qquad L_2 = 1, \qquad L_3 = 1, \qquad L_4 = 3, \qquad L_5 = 7$$

and

$$M^1 = 2, \qquad M^2 = 2, \qquad M^3 = 2, \qquad M^4 = 2^2, \qquad M^5 = 2^3.$$

Each block has the following initial configuration, respectively,

$$503 \quad 200 \quad 402 \quad 1\,020\,104 \quad 400\,020\,205\,000\,204.$$

Therefore we have

$$\begin{cases} Q_0^1 = 1 \\ Q_1^1 = 1 \end{cases} \quad \begin{cases} Q_0^2 = 2 \\ Q_1^2 = 1 \end{cases} \quad \begin{cases} Q_0^3 = 2 \\ Q_1^3 = 1 \end{cases} \quad \begin{cases} Q_0^4 = 1 \\ Q_1^4 = 1 \end{cases} \quad \begin{cases} Q_0^5 = 2 \\ Q_1^5 = 1 \end{cases}$$

and

$$P = \operatorname{lcm}\left(\frac{2}{1 \times 1}, \frac{2}{2 \times 1}, \frac{2}{2 \times 1}, \frac{2^2}{1 \times 1}, \frac{2^3}{2 \times 1}\right) = 2^2.$$

Thus the fundamental period of the initial configuration in the rule (19) is

$$\frac{51 \times 2^2}{\gcd(51, 1 \times 2^2)} = 204.$$

Table 2. Classification of all reversible rules of ECA with periodic boundaries.

Type	Rule
Trivial	15 51 85 170 204 240
Additive	105 150
Linearizable	154 166 180 210
Unknown	45 75 89 101

Finally, we mention conserved quantities. Consider a configuration of size $N \not\equiv 0 \pmod{r-1}$ which contains m blocks of size $(r-1)L_1+1, (r-1)L_2+1, \ldots, (r-1)L_m+1$ as above. Let the size of the boundary wall adjacent to the block of size $(r-1)L_i+1$ be $U_i - 1$ $(i = 1, 2, \ldots, m)$, where $U_i \in \mathbb{N}$. Since both $(r-1)L_i+1$ and $U_i - 1$ never change in the time evolution, the $2 \times m$ matrix

$$\begin{pmatrix} L_1 & L_2 & \cdots & L_m \\ U_1 & U_2 & \cdots & U_m \end{pmatrix} \tag{20}$$

is a conserved quantity.

For a fixed $N \not\equiv 0 \pmod{r-1}$, the number of the conserved quantities (20) is counted as follows. Let Δ be a partition of N which consists of m_Δ numbers equally more than r: $\Delta = (\lambda_1, \ldots, \lambda_{m_\Delta})$, where $\lambda_1, \ldots, \lambda_{m_\Delta} \geqslant r$. Then the number of the matrix (20) is

$$\sum_{\Delta} \prod_{i=1}^{m_\Delta} \left(\left\lceil \frac{\lambda_i}{r-1} \right\rceil - 1 \right),$$

where $\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$ is the ceiling function and the sum ranges over possible partitions Δ of N which consists of numbers equally more than r.

5. Discussion

We showed that the initial value problem for the reversible rule 154 of ECA with periodic boundaries can be reduced to an initial-boundary value problem for a linear system on the finite commutative ring $\mathbb{Z}_2$. Thus the initial value problem for the rule 154 with periodic boundaries can be solved, and hence it can be considered integrable. The congruent rules 166, 180 and 210 with respect to the reflection and the conjugation are also reduced to some linear systems, respectively.

The reversible ECA are classified into four types: trivial, additive, linearizable and unknown (see table 2). The local transitions of the rules 45, 75, 89 and 101 of unknown type are compositions of the local transitions of the linearizable rules 210, 180, 166 and 154 and the 0–1 exchange, respectively. However, their behaviours in the time evolution are much different from the linearizable ones. For instance, the fundamental periods of generic configurations in these rules are much longer than those in the linearizable ones. Making a thorough investigation of these rules is a further problem.

We obtained a family of linearizable cellular automata $\mathcal{A}_r^{(n)}$, each of which can be regarded as a generalization of the reversible rule 154 to arbitrary $n \in \mathbb{N}$ and $r \in \mathbb{N}$. Each member of this family can be reduced to a linear system arising from the permutation (9) and the fundamental period of arbitrary configuration is exactly computed (see proposition 3). As in the case of the rule 154, the congruent rules can also be generalized to linearizable cellular automata. We finally remark that permutations other than (9) do not always lead to reversible systems. For example, let $n = 3$ and $r = 3$. Consider a rule in which sequences of numbers 00, 10 and 20 act as the following permutations when they hit a cell from the left, respectively,

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then there exist irreversible configurations of size $N \equiv 1 \pmod 2$ in this rule. Actually, since both configurations 100 and 201 of size $N = 3$ evolve into a configuration 201, the configuration 201 is not reversible.

Acknowledgment

This work is partially supported by grants-in-aid for scientific research, Japan society for the promotion of science (JSPS).

References

[1] Wolfram S 1983 *Rev. Mod. Phys.* **55** 601–44
[2] Wolfram S 1984 *Physica* D **10** 1–35
[3] Wolfram S 1985 *Phys. Scr.* T **9** 170–83
[4] Martin O, Odlyzko A and Wolfram S 1984 *Commun. Math. Phys.* **93** 219–59
[5] Takesue S 1989 *J. Stat. Phys.* **56** 371–402
[6] Hattori T and Takesue S 1991 *Physica* D **49** 295–322
[7] Takahashi D and Satsuma J 1990 *J. Phys. Soc. Japan* **59** 3514–19
[8] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 *Phys. Rev. Lett.* **76** 3247–50
[9] Torii M, Takahashi D and Satsuma J 1996 *Physica* D **92** 209–20
[10] Matsukidaira J, Satsuma J, Takahashi D, Tokihiro T and Torii M 1997 *Phys. Lett.* A **225** 287–95
[11] Takahashi D and Matsukidaira J 1997 *J. Phys. A: Math. Gen.* **30** L733–9
[12] Hatayama G, Hikami K, Inoue R, Kuniba A, Takagi T and Tokihiro T 2001 *J. Math. Phys.* **42** 274–308
[13] Yura F and Tokihiro T 2002 *J. Phys. A: Math. Gen.* **35** 3787–801
[14] Yoshihara D, Yura F and Tokihiro T 2003 *J. Phys. A: Math. Gen.* **36** 99–121
[15] Nobe A and Yura F 2004 *J. Phys. A: Math. Gen.* **37** 5789–804
[16] Kuniba A and Sakamoto R 2006 *Preprint* nlin/0611046v1
[17] Mada J, Idzumi M and Tokihiro T 2006 *J. Phys. A: Math. Gen.* **39** L617–23
[18] Mendelsohn N S 1970 *Combinatorial Theory and its Applications II: Proc. Coll. (Balatonfüred, 1969)* (Amsterdam: North-Holland) pp 783–99
[19] Nasu M 1978 *Math. Syst. Theory* **11** 327–51
[20] Jen E 1987 *Complex Syst.* **1** 1045–62
[21] Godsil C and Royle G 2001 *Algebraic Graph Theory: Graduate Texts in Mathematics* vol 207 (New York: Springer)
[22] Knuth D 1997 *Seminumerical Algorithms: The Art of Computer Programming* vol 2 (Boston: Addison-Wesley)